Boundedness of commutators generated by m-th Calderón-Zygmund type singular integrals and local Campanato functions on generalized local Morrey spaces

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Abstract Let T_m be the m-th Calderón-Zygmund type singular integral. In the paper, we consider the boundedness of T_m on the generalized product local Morrey spaces $LM^{\{x_0\}}_{p_1,\varphi_1} \times LM^{\{x_0\}}_{p_2,\varphi_2} \times \cdots \times LM^{\{x_0\}}_{p_m,\varphi_m}$. And, the boundedness of the commutators of T_m with local Campanato functions is obtained, also.

Key words m-th Calderón-Zygmud type singular integral, commutator, local Campanato function, generalized local Morrey space

1 Introduction

In recent years, the multilinear singular integrals have been attracting attention and great developments have been achieved (see [1-11]). The study for the multilinear singular integrals is motivated not only by a mere quest to generalize the theory of linear operators but also by their natural appearance in analysis.

Meanwhile, the commutators generated by the multilinear singular integral and BMO functions or Lipschitz functions also attract much attention, since the commutator is more singular than the singular integral operator itself.

Moreover, the classical Morrey space $M_{p,\lambda}$ were first introduced by Morrey in [11] to study the local behavior of solutions to second order elliptic partial differential equations. In [12], the authors studied the boundedess of the multilinear Calderón-Zygmund singular integral on the classical Morrey space $M_{p,\lambda}$. And, in [13], the authors introduced the local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi}$, and they also studied the boundedness of the homogeneous singular integrals with rough kernel on these spaces.

Motivated by the works of [12, 13], we are going to consider the boundedness of the multilinear Calderón-Zygmund singular integral and its commutator on the local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi}$

Now, let us give some related notations.

We are going to be working in $\mathbb{R}^n$. Let $m \in \mathbb{N}$ and $K(y_0, y_1, \ldots, y_m)$ be a function defined away from the diagonal $y_0 = y_1 =, \ldots, = y_m$ in $(\mathbb{R}^n)^{m+1}$. Let T_m be a multilinear

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operator which was initially defined on the m-fold product of Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and take its values in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ and such that for K, the integral representation below is valid:

$$T_m(\vec{f})(x) = T_m(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m) dy_1 \ldots dy_m, \tag{1.1}$$

whenever f_i, $i = 1, \ldots, m$, are smooth functions with compact support and $x \notin \cap_{i=1}^m \mathrm{supp} f_i$.

Moreover, if the kernel K satisfies the following size and smoothness estimates:

$$|K(y_0, y_1, \ldots, y_m)| \leq \frac{C}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}}, \tag{1.2}$$

for all $(y_0, y_1, \ldots, y_m) \in (\mathbb{R}^n)^{m+1}$ away from the diagonal;

$$|K(y_0, \ldots, y_i, \ldots, y_m) - K(y_0, \ldots, y_i', \ldots, y_m)| \leq \frac{C|y_i - y_i'|^\epsilon}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\epsilon}}, \tag{1.3}$$

for some $C > 0$ and $\epsilon > 0$, whenever $0 \leq j \leq m$ and $|y_i - y_i'| \leq 1/2 \max_{0 \leq k \leq m} |y_i - y_k|$, then the kernel is called a m-th Calderón-Zygmund kernel and the collection of such functions is denoted by $m - CZK(C, \epsilon)$. Let T_m be as in (1.1) with a $m - CZK(C, \epsilon)$ kernel, then T_m is called a m-th Calderón-Zygmund type singular integral and the collection of these operators is denoted by $m - CZO$.

Now, we define the commutators generated by the m-th multilinear Calderón-Zygmund type singular integral as follows.

Let $\vec{b} = (b_1, \ldots, b_m)$ be a finite family of locally integrable functions, then the commutators generated by the m-th Calderón-Zygmund type singular integral and $\vec{b}$ is defined by:

$$T_m^{\vec{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \prod_{i=1}^m (b_i(x) - b_i(y_i)) f_i(y_i) dy_1 \ldots dy_m.$$

In the following, we will establish the boundedness of T_m on generalized product local Morrey spaces. And, we also consider the boundedness of the commutators generated by the m-th Calderón-Zygmund type singular integral T_m and the local Campanato function on generalized product local Morrey spaces.

2 Some notations and lemmas

Definition 2.1[13] Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p \leq \infty$. For any fixed $x_0 \in \mathbb{R}^n$, a function $f \in L_{loc}^q$ is said to belong to the local Morrey space, if

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \varphi^{-1}(x_0, r) |B(x_0, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x_0,r))} < \infty.$$

And, we denote

$$LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n) = \{f \in L_{loc}^q(\mathbb{R}^n) : \|f\|_{LM_{p,\varphi}^{\{x_0\}}} < \infty\}.$$

According to this definition, we recover the local Morrey space$LM_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$.

Definition 2.2[13] Let $1 \leq q < \infty$ and $0 \leq \lambda < 1/n$. A function $f \in L_{loc}^q(\mathbb{R}^n)$ is said to belong to the space $LC_{q,\lambda}^{\{x_0\}}$ (local Campanato space), if

$$\|f\|_{LC_{q,\lambda}^{\{x_0\}}} = \sup_{r>0}\left(\frac{1}{|B(x_0,r)|^{1+\lambda q}}\int_{B(x,r)}|f(y) - f_{B(x_0,r)}|^q dy\right)^{1/q} < \infty,$$

where

$$f_{B(x_0,r)} = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(y)dy.$$

Define

$$LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \{f \in L_{loc}^q(\mathbb{R}^n) : \|f\|_{LC_{q,\lambda}^{\{x_0\}}} < \infty\}.$$

Remark.[13] Note that, the central BMO space $CBMO_q(\mathbb{R}^n) = LC_{q,0}^{\{0\}}(\mathbb{R}^n)$, $CBMO_q^{\{x_0\}}(\mathbb{R}^n) = LC_{q,0}^{\{x_0\}}(\mathbb{R}^n)$, and $BMO_q(\mathbb{R}^n) \subset \bigcap_{q>1} CBMO_q^{\{x_0\}}(\mathbb{R}^n)$. Moreover, one can imagine that the behavior of $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$, since there is no analogy of the John-Nirenberg inequality of BMO for the space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$.

Lemma 2.1 Let $1 < q < \infty$, $0 < r_2 < r_1$ and $b \in LC_{q,\lambda}^{\{x_0\}}$, then

$$\left(\frac{1}{|B(x_0,r_1)|^{1+\lambda q}}\int_{B(x_0,r_1)}|b(x) - b_{B(x_0,r_2)}|^q dx\right)^{1/q} \leq C\left(1 + \ln\frac{r_1}{r_2}\right)\|b\|_{LC_{q,\lambda}^{\{x_0\}}}. \tag{2.1}$$

And, from this inequality, we have

$$|b_{B(x_0,r_1)} - b_{B(x_0,r_2)}| \leq C\left(1 + \ln\frac{r_1}{r_2}\right)|B(x_0,r_1)|^{\lambda}\|b\|_{LC_{q,\lambda}^{\{x_0\}}}. \tag{2.2}$$

In this section, we are going to use the following statement on the boundedness of the weighted Hardy operator:

$$H_w g(t) := \int_t^\infty g(s)w(s)ds,\ 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.

Lemma 2.2[14, 15] Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality

$$ess \sup_{t>0} v(2t) H_w g(t) \leq C ess \sup_{t>0} v_1(t) g(t) \tag{2.3}$$

holds for some $C>0$ and all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B: ess \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{ess \sup_{s<\tau<\infty} v_1(\tau)} ds < \infty.$$

Moreover, if $\tilde{C}$ is the minimum value of C in (2.3), then $\tilde{C} = B$.

Lemma 2.3[2] Let T_m be a $m - CZO$. Suppose that $1 \leq p_1, \cdots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. If $p_i > 1, i = 1, \cdots, m$, then there exists a constant $C > 0$, such that

$$\|T_m \vec{f}\|_{L^p} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}}.$$

3 M-th Calderón-Zygmund type singular integral operator on generalized product local Morrey space

Theorem 3.1 Let $x_0 \in \mathbb{R}^n$, $1 < p, p_1, p_2, \ldots, p_m < \infty$, such that $1/p = 1/p_1 + 1/p_2 + \cdots + p_m$. Then the inequality

$$\|T_m(\vec{f})\|_{L^p(B(x_0,r))} \lesssim r^{n/p} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0,r))} t^{-n/p-1} dt$$

holds for any ball $B(x_0, r)$ and all $f_i \in L^{p_i}_{loc}(\mathbb{R}^n)$, $i = 1, 2, \ldots, m$.

Proof. Without loss of generality, it is suffice to show that the conclusion holds for $T_2(f_1, f_2)$.

Let $B = B(x_0, r)$. And, we write $f_1 = f_1^0 + f_1^\infty$ and $f_2 = f_2^0 + f_2^\infty$, where $f_i^0 = f_i \chi_{2B}$, $f_i^\infty = f_i \chi_{(2B)^c}$, for $i = 1.2$. Thus, we have

$$\begin{aligned}
& \|T_2(f_1, f_2)\|_{L^p(B(x_0,r))} \\
\leq\ & \|T_2(f_1^0, f_2^0)\|_{L^p(B)} + \|T_2(f_1^0, f_2^\infty)\|_{L^p(B)} + \|T_2(f_1^\infty, f_2^0)\|_{L^p(B)} + \|T_2(f_1^\infty, f_2^\infty)\|_{L^p(B)} \\
=:\ & I + II + III + IV.
\end{aligned}$$

Using the L^p boundedness of T_2(Lemma 2.3), we have

$$\begin{aligned}
I \lesssim\ & \|f_1\|_{L^{p_1}(2B)} \|f_2\|_{L^{p_2}(2B)} \\
\lesssim\ & r^{\frac{n}{p}} \|f_1\|_{L^{p_2}(2B)} \|f_2\|_{L^{p_2}(2B)} \int_{2r}^\infty \frac{dt}{t^{\frac{n}{p}+1}} \\
\leq\ & r^{\frac{n}{p}} \int_{2r}^\infty \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.
\end{aligned} \tag{3.1}$$

Moreover, when $x \in B(x_0, r)$ and $y \in (2B)^c$, we have

$$\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|. \tag{3.2}$$

Then, it follows from (1.2) that

$$\begin{aligned} |T_2(f_1^0, f_2^\infty)(x)| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f_1^0(y_1)||f_2^\infty(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}} dy_1 dy_2 \\ &\lesssim \int_{2B} |f_1(y_1)| dy_1 \int_{(2B)^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^{2n}} dy_2 \\ &\lesssim \int_{2B} |f_1(y_1)| dy_1 \int_{(2B)^c} |f_2(y_2)| \left[\int_{|x_0-y_2|}^\infty \frac{dt}{t^{2n+1}}\right] dy_2 \\ &\lesssim \|f_1\|_{L^{p_1}(2B)} |2B|^{1-1/p_1} \int_{2r}^\infty \|f_2\|_{L^{p_2}(B(x_0,t))} |B(x_0,t)|^{1-1/p_2} \frac{dt}{t^{2n+1}} \\ &\lesssim \int_{2r}^\infty \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} \frac{dt}{t^{n/p+1}}, \end{aligned} \tag{3.3}$$

where $1/p = 1/p_1 + 1/p_2$.

Thus,

$$II = \|T_2(f_1^0, f_2^\infty)\|_{L^p(B)} \lesssim r^{n/p} \int_{2r}^\infty \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} \frac{dt}{t^{n/p+1}}. \tag{3.4}$$

Similarly, we have

$$III = \|T_2(f_1^\infty, f_2^0)\|_{L^p(B)} \lesssim r^{n/p} \int_{2r}^\infty \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} \frac{dt}{t^{n/p+1}}.$$

Moreover, similar to the estimate of (3.3), we have

$$\begin{aligned} |T_2(f_1^\infty, f_2^\infty)(x)| &\lesssim \int_{(2B)^c}\int_{(2B)^c} \frac{|f_1(y_1)||f_2(y_2)|}{(|x_0-y_1|+|x_0-y_2|)^{2n}} dy_1 dy_2 \\ &\lesssim \int_{(2B)^c}\int_{(2B)^c} |f_1(y_1)||f_2(y_2)| dy_1 dy_2 \int_{|x_0-y_1|+|x_0-y_2|}^\infty \frac{dt}{t^{2n+1}} \\ &\lesssim \int_{2r}^\infty \left[\int_{B(x_0,t)} |f_1(y_1)| dy_1 \int_{B(x_0,t)} |f_2(y_2)| dy_2\right] \frac{dt}{t^{2n+1}} \\ &\lesssim \int_{2r}^\infty \|f\|_{L^{p_1}(B(x_0,t))} \|f\|_{L^{p_1}(B(x_0,t))} |B(x_0,t)|^{2-(1/p_1+1/p_2)} \frac{dt}{t^{2n+1}} \\ &\lesssim \int_{2r}^\infty \|f\|_{L^{p_1}(B(x_0,t))} \|f\|_{L^{p_1}(B(x_0,t))} \frac{dt}{t^{n/p+1}}. \end{aligned}$$

Thus,

$$IV = \|T_2(f_1^\infty, f_2^\infty)\|_{L^p(B)} \lesssim r^{n/p} \int_{2r}^\infty \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} \frac{dt}{t^{n/p+1}}. \tag{3.5}$$

Combining the above estimates, we obtain

$$\|T_2(f_1,f_2)\|_{L^p(B)} \lesssim r^{n/p}\int_{2r}^{\infty}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}\frac{dt}{t^{n/p+1}}.$$

Theorem 3.2 Let $x_0\in\mathbb{R}^n$, $1<p,p_1,p_2,\ldots,p_m<\infty$ such that $1/p=1/p_1+1/p_2+\cdots+p_m$. If functions φ, $\varphi_i:\mathbb{R}^n\times(0,\infty)\to(0,+\infty)$, $(i=1,2,\cdots,m)$ satisfy the condition

$$\int_r^{\infty}\frac{\operatorname{ess}\inf\limits_{t<s<\infty}\prod\limits_{i=1}^{m}\varphi_i(x_0,s)s^{n/p}}{t^{n/p+1}}dt\le C\psi(x_0,r), \tag{3.6}$$

where constant $C>0$ doesn't depend on r. Then the operator T_m is bounded from the product space $LM^{\{x_0\}}_{p_1,\varphi_1}\times LM^{\{x_0\}}_{p_2,\varphi_2}\times\cdots\times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\psi}$. Moreover, the following inequality

$$\|T_m(\vec{f})\|_{LM^{\{x_0\}}_{p,\psi}}\lesssim\prod_{i=1}^{m}\|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}.$$

holds.

Proof. Taking $v_1(r)=\prod\limits_{i=1}^{m}\varphi_i^{-1}(x_0,r)r^{-n/p}$, $v_2(r)=\psi^{-1}(x_0,r)$, $g(r)=\prod\limits_{i=1}^{m}\|f_i\|_{L^{p_i}(B(x_0,r))}$ and $w(r)=r^{-n/p-1}$, then we have

$$ess\sup_{t>0}v_2(t)\int_t^{\infty}\frac{w(s)ds}{ess\sup\limits_{s<\tau<\infty}v_1(\tau)}<\infty.$$

Thus, by Lemma 2.2, we have

$$ess\sup_{t>0}v_2(t)H_wg(t)\le Cess\sup_{t>0}v_1(t)g(t). \tag{3.7}$$

Therefore, from Theorem 3.1 and (3.7), it follows that

$$\begin{aligned}
&\|T_m(\vec{f})\|_{LM^{\{x_0\}}_{p,\psi}}\\
=\;&\sup_{r>0}\psi^{-1}(x_0,r)|B(x_0,r)|^{-1/p}\|T_m(\vec{f})\|_{L^p(B(x_0,r))}\\
\lesssim\;&\sup_{r>0}\psi^{-1}(x_0,r)|B(x_0,r)|^{-1/p}r^{n/p}\int_{2r}^{\infty}\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(B(x_0,t))}t^{-n/p-1}dt\\
\lesssim\;&\sup_{r>0}\prod_{i}^{m}\varphi_i^{-1}(x_0,r)r^{-n/p}\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(B(x_0,r))}\\
\lesssim\;&\sup_{r>0}\prod_{i=1}^{m}\varphi_i^{-1}(x_0,r)r^{-n/p_i}\|f_i\|_{L^{p_i}(B(x_0,r))}\\
=\;&\prod_{i=1}^{m}\|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}.
\end{aligned}$$

4 Commutators generated by m-th Calderón Zygmund type singular integral operators and local Campanato functions

Theorem 4.1 Let $x_0 \in \mathbb{R}^n$, $1 < p$, p_i, $q_i < \infty (i = 1, 2, \ldots, m)$ such that $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m + 1/p_1 + 1/q_2 + \cdots + 1/q_m$ and $b_i \in LC^{\{x_0\}}_{q_i,\lambda_i}$ for $0 < \lambda_i < 1/n$, $i = 1, 2, \cdots, m$. Then the inequality

$$\|T^{\vec{b}}_m(\vec{f})\|_{L^p(B(x_0,r))} \lesssim \prod_{i=1}^m \|b_i\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{n/p} \int_{2r}^{\infty} \left(1+\ln \frac{t}{r}\right)^m t^{n\sum\limits_{i=1}^m \lambda_i - n\sum\limits_{i=1}^m 1/p_i - 1} \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ and all $f_i \in L^{p_i}_{loc}(\mathbb{R}^n)$, $i = 1, 2, \ldots, m$.

Proof. Without loss of generality, it is suffice for us to show that the conclusion holds for $m = 2$.

Let $B = B(x_0, r)$, $f_1 = f_1^0 + f_1^\infty$ and $f_2 = f_2^0 + f_2^\infty$, where f_i^0 and f_i^∞ are as in the proof of Theorem 3.1, for $i = 1.2$. Thus, we have

$$\begin{aligned}
& T_2^{(b_1,b_2)}(f_1, f_2)(x) \\
= \ & T_2^{(b_1,b_2)}(f_1^0, f_2^0)(x) + T_2^{(b_2,b_2)}(f_1^0, f_2^\infty)(x) + T_2^{(b_1,b_2)}(f_1^\infty, f_2^0)(x) + T_2^{(b_1,b_2)}(f_1^\infty, f_2^\infty)(x).
\end{aligned}$$

So,

$$\begin{aligned}
& \|T_2^{(b_1,b_2)}(f_1, f_2)\|_{L^p(B)} \\
& \leq \|T_2^{(b_1,b_2)}(f_1^0, f_2^0)\|_{L^p(B)} + \|T_2^{(b_1,b_2)}(f_1^0, f_2^\infty)\|_{L^p(B)} \\
& \quad + \|T_2^{(b_1,b_2)}(f_1^\infty, f_2^0)\|_{L^p(B)} + \|T_2^{(b_1,b_2)}(f_1^\infty, f_2^\infty)\|_{L^p(B)} \\
& =: I + II + III + IV.
\end{aligned}$$

Let us estimate I, II, III and IV, respectively.

Since,

$$\begin{aligned}
& (b_1(x) - b_1(y))(b_2(x) - b_2(y)) \\
& = (b_1(x) - (b_1)_B)(b_2(x) - (b_2)_B) - (b_1(x) - (b_1)_B)(b_2(y) - (b_2)_B) \\
& \quad - (b_1(y) - (b_1)_B)(b_2(x) - (b_2)_B) + (b_1(y) - (b_1)_B)(b_2(y) - (b_2)_B).
\end{aligned} \tag{4.1}$$

Then,

$$\begin{aligned}
& \|T_2^{(b_1,b_2)}(f_1^0, f_2^0)\|_{L^p(B)} \\
= \ & \|(b_1 - (b_1)_B)(b_2 - (b_2)_B)T_2(f_1^0, f_2^0)\|_{L^p(B)} + \|(b_1 - (b_1)_B)T_2(f_1^0, (b_2 - (b_2)_B)f_2^0)\|_{L^p(B)} \\
& + \|(b_2 - (b_2)_B)T_2((b_1 - (b_1)_B)f_1^0, f_2^0)\|_{L^p(B)} + \|T_2((b_1 - (b_1)_B)f_1^0, (b_2 - (b_2)_B)f_2^0)\|_{L^p(B)} \\
=: \ & I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.2}$$

Let $1<\bar{p},\bar{q}<\infty$, such that $1/\bar{p}=1/p_1+1/p_2$ and $1/\bar{q}=1/q_1+1/q_2$. Then, using the Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}
I_1 &\lesssim \|(b_1-(b_1)_B)(b_2-(b_2)_B\|_{L^{\bar{q}}(B)}\|T_2(f_1^0,f_2^0)\|_{L^{\bar{p}}(B)}\\
&\lesssim \|b_1-(b_1)_B\|_{L^{q_2}(B)}\|b_2-(b_2)_B\|_{L^{q_2}(B)}\|f_1\|_{L^{p_1}(2B)}\|f_2\|_{L^{p_1}(2B)}\\
&\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|b_2-(b_2)_B\|_{L^{q_2}(B)}r^{(1/p_1+1/p_2)n}\\
&\quad\times\int_{2r}^{\infty}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}\frac{dt}{t^{(1/p_1+1/p_2)n+1}}\\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{n/p}\\
&\quad\times\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}dt.
\end{aligned}\tag{4.3}$$

Let $1<\tau<\infty$, such that $1/p=1/q_1+1/\tau$. Then similarly to the estimate of (4.3), we have

$$\begin{aligned}
I_2 &\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|T_2(f_1^0,(b_2-(b_2)_B)f_2^0)\|_{L^{\tau}(B)}\\
&\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|f_1^0\|_{L^{p_1}(\mathbb{R}^n)}\|(b_2-(b_2)_{2B})f_2^0)\|_{L^{s}(\mathbb{R}^n)}\\
&\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|b_2-(b_2)_B\|_{L^{q_2}(2B)}\|f_1\|_{L^{p_1}(2B)}\|f_2\|_{L^{p_2}(2B)},
\end{aligned}\tag{4.4}$$

where $1<s<\infty$, such that $1/s=1/p_2+1/q_2=1/\tau-1/p_1$.

From Lemma 2.1, it is easy to see that

$$\|b_i-(b_i)_B\|_{L^{q_i}(B)}\le Cr^{n/q_i+n\lambda_i}\|b_i\|_{LC^{\{x_0\}}_{q_i,\lambda_i}},$$

and

$$\|b_i-(b_i)_B\|_{L^{q_i}(2B)}\le\|b_i-(b_i)_{2B}\|_{L^{q_i}(2B)}+\|(b_i)_B-(b_i)_{2B}\|_{L^{q_i}(2B)}\le Cr^{n/q_i+n\lambda_i}\|b_i\|_{LC^{\{x_0\}}_{q_i,\lambda_i}},\tag{4.5}$$

for $i=1,2$.

Then,

$$\begin{aligned}
I_2 &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{n/p}\\
&\quad\times\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}dt.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_3 &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{n/p}\\
&\quad\times\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}dt.
\end{aligned}$$

Moreover, let $1<\tau_1,\tau_2<\infty$, such that $1/\tau_1=1/p_1+1/q_1$ and $1/\tau_2=1/p_2+1/q_2$. It is easy to see that $1/p=1/\tau_1+1/\tau_2$. Then by Lemma 2.3, Hölder's inequality and (4.5),

we obtain

$$\begin{aligned}
I_4 &\lesssim \|(b_1-(b_1)_B)f_1^0\|_{L^{\tau_1}(\mathbb{R}^n)}\|(b_2-(b_2)_B)f_2^0\|_{L^{\tau_2}(\mathbb{R}^n)}\\
&\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(2B)}\|b_2-(b_2)_B\|_{L^{q_2}(2B)}\|f_1\|_{L^{p_1}(2B)}\|f_2\|_{L^{p_2}(2B)}\\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{n/p}\\
&\quad\times\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}dt.
\end{aligned} \tag{4.6}$$

Therefore, combining the estimates of I_1, I_2, I_3 and I_4, we have

$$\begin{aligned}
I &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{n/p}\\
&\quad\times\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}dt.
\end{aligned}$$

Let us estimate II.

It's analogues to (4.2), we have

$$\begin{aligned}
&\|T_2^{(b_1,b_2)}(f_1^0,f_2^\infty)\|_{L^p(B)}\\
=\;&\|(b_1-(b_1)_B)(b_2-(b_2)_B)T_2(f_1^0,f_2^\infty)\|_{L^p(B)}+\|(b_1-(b_1)_B)T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L^p(B)}\\
&+\|(b_2-(b_2)_B)T_2((b_1-(b_1)_B)f_1^0,f_2^\infty)\|_{L^p(B)}+\|T_2((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L^p(B)}\\
=:\;&II_1+II_2+I_3+II_4.
\end{aligned} \tag{4.7}$$

Let $1<\bar{p},\bar{q}<\infty$, such that $1/\bar{p}=1/p_1+1/p_2$ and $1/\bar{q}=1/q_1+1/q_2$. Then, using the Hölder's inequality and (3.4), we have

$$\begin{aligned}
II_1 &\lesssim \|(b_1-(b_1)_B)(b_2-(b_2)_{2B}\|_{L^{\bar{q}}(B)}\|T_2(f_1^0,f_2^\infty)\|_{L^{\bar{p}}(B)}\\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{(1/q_1+1/q_2)n+(\lambda_1+\lambda_2)n} r^{(1/p_1+1/p_2)n}\\
&\quad\times\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2 t^{-(1/p_1+1/p_2)n-1}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}dt\\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{n/p}\\
&\quad\times\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}dt.
\end{aligned} \tag{4.8}$$

Moreover, using (1.2) and (3.2), we have

$$\begin{aligned}
&|T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)(x)|\\
&\lesssim \int_{2B}|f_1(y_1)|dy_1\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n}}dy_2.
\end{aligned}$$

It's obvious that

$$\int_{2B}|f_1(y_1)|dy_1\lesssim\|f_1\|_{L^{p_1}(2B)}|2B|^{1-1/p_1}, \tag{4.9}$$

and

$$
\begin{aligned}
&\int_{(2B)^c} \frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n}} dy_2 \\
&\lesssim \int_{(2B)^c} |b_2(y_2)-(b_2)_B||f_2(y_2)| \left[\int_{|x_0-y_2|}^{\infty} \frac{dt}{t^{2n+1}}\right] dy_2 \\
&\lesssim \int_{2r}^{\infty} \|b_2(y_2)-(b_2)_{B(x_0,t)}\|_{L^{q_2}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} |B(x_0,t)|^{1-(1/p_2+1/q_2)} \frac{dt}{t^{2n+1}} \\
&\quad + \int_{2r}^{\infty} |(b_2)_{B(x_0,t)}-(b_2)_{B(x_0,r)}| \|f_2\|_{L^{p_2}(B(x_0,t))} |B(x_0,t)|^{1-1/p_2} \frac{dt}{t^{2n+1}} \\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^{\infty} |B(x_0,t)|^{1/q_2+\lambda_2} \|f_2\|_{L^{p_2}(B(x_0,t))} |B(x_0,t)|^{1-(1/p_2+1/q_2)} \frac{dt}{t^{2n+1}} \\
&\quad + \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) |B(x_0,t)|^{\lambda_2} \|f_2\|_{L^{p_2}(B(x_0,t))} |B(x_0,t)|^{1-1/p_2} \frac{dt}{t^{2n+1}} \\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{-n+n\lambda_2-n/p_2-1} \|f_2\|_{L^{p_2}(B(x_0,t))} dt.
\end{aligned}
\tag{4.10}
$$

Therefore, from (4.9) and (4.10), it follows that

$$
\begin{aligned}
&|T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)(x)| \\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} \|f_1\|_{L^{p_1}(2B)} |2B|^{1-1/p_1} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{-n+n\lambda_2-n/p_2-1} \|f_2\|_{L^{p_2}(B(x_0,t))} dt \\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{n\lambda_2-(1/p_1+1/p_2)n-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt.
\end{aligned}
$$

Thus, let $1<\tau<\infty$, such that $1/p=1/q_1+1/\tau$, then similarly to the estimate of (4.3), we have

$$
\begin{aligned}
II_2 &= \|(b_1-(b_1)_B)T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L^p(B)} \\
&\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)} \|T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L^\tau(B)} \\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} |B|^{\lambda_1+1/q_1+1/\tau} \\
&\quad \times \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{n\lambda_2-(1/p_1+1/p_2)n-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt \\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{n/p} \\
&\quad \times \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt.
\end{aligned}
\tag{4.11}
$$

Similarly, we have

$$
\begin{aligned}
II_3 &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{n/p} \\
&\quad \times \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt.
\end{aligned}
$$

Let us estimate II_4.

Since,

$$\begin{aligned}&|T_2((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^\infty)(x)|\\&\lesssim \int_{2B}|b_1(y_1)-(b_1)_B||f_1(y_1)|dy_1\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n}}dy_2,\end{aligned}$$

and

$$\int_{2B}|b_1(y_1)-(b_1)_B||f_1(y_1)|dy_1\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}|B|^{\lambda_1+1-1/p_1}\|f_1\|_{L^{p_1}(2B)}. \tag{4.12}$$

Then, by (4.10) and (4.12), we have

$$\begin{aligned}&|T_2((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^\infty)(x)|\\&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^\infty\left(1+\ln\frac{t}{r}\right)^2 t^{n(\lambda_1+\lambda_2)-n(1/p_1+/p_2)-1}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}dt.\end{aligned}$$

Therefore,

$$\begin{aligned}II_4&=\|T_2((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L^p(B)}\\&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{n/p}\\&\quad\times\int_{2r}^\infty\left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}dt.\end{aligned}$$

Combining the estimates of II_1-II_4, we have

$$\begin{aligned}II&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{n/p}\\&\quad\times\int_{2r}^\infty\left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}dt.\end{aligned}$$

Similarly,

$$\begin{aligned}III&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{n/p}\\&\quad\times\int_{2r}^\infty\left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}dt.\end{aligned}$$

For IV, we have

$$\begin{aligned}&\|T_2^{(b_1,b_2)}(f_1^\infty,f_2^\infty)\|_{L^p(B)}\\\le\ &\|(b_1-(b_1)_B)(b_2-(b_2)_B)T_2(f_1^\infty,f_2^\infty)\|_{L^p(B)}+\|(b_1-(b_1)_B)T_2(f_1^\infty,(b_2-(b_2)_B)f_2^\infty)\|_{L^p(B)}\\&\quad+\|(b_2-(b_2)_B)T_2((b_1-(b_1)_B)f_1^\infty,f_2^\infty)\|_{L^p(B)}+\|T_2((b_1-(b_1)_B)f_1^\infty,(b_2-(b_2)_B)f_2^\infty)\|_{L^p(B)}\\=:\ &IV_1+IV_2+IV_3+IV_4.\end{aligned}$$

Let us estimate IV_1, IV_2, IV_3 and IV_4, respectively.

Let $1 < \tau < \infty$, such that $1/p = 1/q_1 + 1/q_2 + 1/\tau$. Then, from Hölder's inequality and (3.5), we get

$$\begin{aligned} IV_1 &\lesssim \|b_1 - (b_1)_B\|_{L^{q_1}(B)} \|b_2 - (b_2)_B\|_{L^{q_2}(B)} \|T_2(f_1^\infty, f_2^\infty)\|_{L^\tau(B)} \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} |B|^{(\lambda_1+\lambda_2)+(1/q_1+1/q_2)+1/\tau} \\ &\quad \times \int_{2r}^\infty \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} t^{-n(/p_1+1/p_2)-1} dt \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{n/p} \\ &\quad \times \int_{2r}^\infty \left(1 + \ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt. \end{aligned}$$

Moreover, by (1.2) and (3.2), we have

$$\begin{aligned} &|T_2(f_1^\infty, (b_2 - (b_2)_B) f_2^\infty)(x)| \\ &\lesssim \int_{(2B)^c} \int_{(2B)^c} \frac{|b_2(y_2) - (b_2)_B||f_1(y_1)||f_2(y_2)|}{(|x_0 - y_1| + |x_0 - y_2|)^{2n}} dy_1 dy_2 \\ &\lesssim \int_{(2B)^c} \int_{(2B)^c} |f_1(y_1)||b_2(y_2) - (b_2)_B||f_2(y_2)| \left[\int_{|x_0-y_1|+|x_0-y_2|}^\infty \frac{dt}{t^{2n+1}}\right] dy_1 dy_2 \\ &\lesssim \int_{2r}^\infty \left[\int_{B(x_0,t)} |f_1(y_1)| dy_1\right] \left[\int_{B(x_0,t)} |b_2(y_2) - (b_2)_B||f_2(y_2)| dy_2\right] \frac{dt}{t^{2n+1}}. \end{aligned}$$

Since,

$$\int_{B(x_0,t)} |f_1(y_1)| dy_1 \lesssim \|f_1\|_{L^{p_1}(B(x_0,t))} t^{n(1-1/p_1)},$$

and

$$\begin{aligned} &\int_{B(x_0,t)} |b_2(y_2) - (b_2)_B||f_2(y_2)| \\ &\lesssim \|b_2 - (b_2)_{B(x_0,t)}\|_{L^{q_2}(B(x_0,t))} \|f_2\|_{L^{p_2}} |B(x_0,t)|^{1-(1/p_2+1/q_2)} \\ &\quad + |(b_2)_{B(x_0,t)} - (b_2)_{B(x_0,r)}| \|f_2\|_{L^{p_2}} |B(x_0,t)|^{1-1/p_2} \\ &\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} |B(x_0,t)|^{1/q_2+\lambda_2} \|f_2\|_{L^{p_2}} |B(x_0,t)|^{1-(1/p_2+1/q_2)} \\ &\quad + \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} \left(1 + \ln\frac{t}{r}\right) |B(x_0,t)|^{\lambda_2} \|f_2\|_{L^{p_2}} |B(x_0,t)|^{1-1/p_2} \\ &\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} \left(1 + \ln\frac{t}{r}\right)^2 t^{n\lambda_2 - n/p_2 + n} \|f_2\|_{L^{p_2}(B(x_0,t))}. \end{aligned}$$

Then,

$$\begin{aligned} &|T_2(f_1^\infty, (b_2 - (b_2)_B) f_2^\infty)(x)| \\ &\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^\infty \left(1 + \ln\frac{t}{r}\right)^2 t^{n\lambda_2-(1/p_1+1/p_2)n-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt. \end{aligned} \tag{4.13}$$

Let $1 < \tau < \infty$, such that $1/p = 1/q_1 + 1/\tau$. Then, from Hölder's inequality and (4.13), we have

$$\begin{aligned} IV_2 &\lesssim \|b_1 - (b_1)_B\|_{L^{q_1}(B)} \|T_2(f_1^\infty, (b_2 - (b_2)_B) f_2^\infty)\|_{L^\tau(B)} \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{n/p} \\ &\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt. \end{aligned}$$

Similarly,

$$\begin{aligned} IV_3 &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{n/p} \\ &\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt. \end{aligned}$$

Similar to the estimate of (4.13), we have

$$\begin{aligned} &|T_2((b_1 - (b_1)_B) f_1^\infty, (b_2 - (b_2)_B) f_2^\infty)(x)| \\ &\lesssim \int_{(2B)^c} \int_{(2B)^c} |b_1(y_1) - (b_1)_B||b_2(y_2) - (b_2)_B||f_1(y_1)||f_2(y_2)| \left[\int_{|x_0-y_1|+|x_0-y_2|}^{\infty} \frac{dt}{t^{2n+1}}\right] dy_1 dy_2 \\ &\lesssim \int_{2r}^{\infty} \left[\int_{B(x_0,t)} |b_1(y_1) - (b_1)_B||f_1(y_1)| dy_1\right] \left[\int_{B(x_0,t)} |b_2(y_2) - (b_2)_B||f_2(y_2)| dy_2\right] \frac{dt}{t^{2n+1}} \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1+\lambda_2)-n(1/p_1+1/p_2)-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt. \end{aligned}$$

Thus,

$$\begin{aligned} IV_4 &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{n/p} \\ &\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt. \end{aligned}$$

Then, from the estimates of $IV_1 - IV_4$, we deduce that

$$\begin{aligned} IV &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{n/p} \\ &\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt. \end{aligned}$$

So, combining the estimates for I, II, III and IV, we have

$$\begin{aligned} &\|T_2^{(b_1,b_2)}(f_1, f_2)\|_{L^p(B)} \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{n/p} \\ &\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-(1/p_1+1/p_2)n-1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt. \end{aligned}$$

Therefore, we complete the proof of Theorem 4.1.

Theorem 4.2 Let $x_0 \in \mathbb{R}^n$, $1 < p, p_i, q_i < \infty$, for $i = 1, 2, \ldots, m$ such that $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_n + 1/p_1 + 1/q_2 + \cdots + 1/q_n$. Suppose that $0 < \lambda_i < 1/n$ such that $b_i \in LC^{\{x_0\}}_{q_i,\lambda_i}$, for $0 < \lambda_i < 1/n$, $i = 1, 2, \cdots, m$. If functions φ, $\varphi_i : \mathbb{R}^n \times (0, \infty) \to (0, +\infty), (i = 1, 2, \cdots, m)$ satisfy the condition

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right)^m t^{n\sum\limits_{i=1}^m \lambda_i - n\sum\limits_{i=1}^m 1/p_i - 1} \operatorname{ess} \inf_{t<s<\infty} \prod_{i=1}^m \varphi_i(x_0, s) s^{n/p_i} dt \le C\psi(x_0, r),$$

where constant $C > 0$ doesn't depend on r. Then the operator $T^{\vec{b}}_m$ is bounded from product space $LM^{\{x_0\}}_{p_1,\varphi_1} \times LM^{\{x_0\}}_{p_2,\varphi_2} \times \cdots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\psi}$. Moreover, the inequality

$$\|T^{\vec{b}}_m(\vec{f})\|_{LM^{\{x_0\}}_{p,\psi}} \lesssim \prod_{i=1}^m \|b_i\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} \prod_{i=1}^m \|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}.$$

holds.

Proof. Taking $v_1(t) = \prod\limits_{i=1}^m \varphi_i^{-1}(x_0, t) t^{-n/p_i}$, $v_2(t) = \psi^{-1}(x_0, t)$, $g(t) = \prod\limits_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0,t))}$ and $w(t) = (1 + \ln \frac{t}{r})^m t^{n\sum\limits_{i=1}^m \lambda_i - n\sum\limits_{i=1}^m 1/p_i - 1}$, then we have

$$ess \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{ess \sup\limits_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Thus, by Lemma 2.2, we have

$$ess \sup_{t>0} v_2(t) H_w g(t) \le C ess \sup_{t>0} v_1(t) g(t).$$

So,

$$\begin{aligned}
& \|T^{\vec{b}}_m(\vec{f})\|_{LM^{\{x_0\}}_{p,\psi}} \\
= & \sup_{r>0} \psi^{-1}(x_0, r) |B(x_0, r)|^{-1/p} \|T_m(\vec{f})\|_{L^p(B(x_0,r))} \\
\lesssim & \prod_{i=1}^m \|b_i\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} \sup_{r>0} \psi^{-1}(x_0, r) \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^m t^{n\sum\limits_{i=1}^m \lambda_i - n\sum\limits_{i=1}^m 1/p_i - 1} \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0,t))} dt \\
\lesssim & \prod_{i=1}^m \|b_i\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} \sup_{r>0} \prod_{i=1}^m \varphi_i^{-1}(x_0, r) r^{-n/p_i} \|f_i\|_{L^{p_i}(B(x_0,r))} \\
= & \prod_{i=1}^m \|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}.
\end{aligned}$$

Thus we complete the proof of Theorem 4.2.

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